Problem 1: Kramers' relation

Kramers' relation, named after the Dutch physicist Hans Kramers, is a relationship between expectation values of *nearby* powers of r for the hydrogen-like atom:

$$\frac{s+1}{n^2}\left\langle r^s\right\rangle-(2s+1)a\left\langle r^{s-1}\right\rangle+\frac{s}{4}\left[(2l+1)^2-s^2\right]a^2\left\langle r^{s-2}\right\rangle=0 \tag{1}$$

Derive the Kramers' relation. Start with writing out the radial Schrödinger equation for a hydrogen-like atom:

$$\frac{d^2u}{dr^2}=\left(\frac{l(l+1)}{r^2}-\frac{2}{ar}+\frac{1}{n^2a^2}\right)u, \tag{2}$$

where $u=rR$ and $a=\hbar^2/(mZe^2)$. Multiply both sides by r^s, and by u^*, and integrate. The left-hand-side integrate by parts in two steps procedure.

Problem 2: Spin-orbit splitting

Calculate the value of spin-orbit splitting of the hydrogen atom for $2p$ state. Give an order of magnitude in eV (is it 10^{-6}, 10^{-4}, 10^{-2}, 10^{0} eV)?

Problem 3: Hyperfine-structure spectrum of hydrogen atom

The ground state of hydrogen atom is $1s$. When examined very closely, it is found that the level is split into two levels. Explain why this splitting takes place. Estimate numerically the energy difference between these two levels. Give the answer in eV.

Problem 4: Muonic hydrogen

A muon is a particle identical to an electron except its mass is about 200 times larger. A muonic hydrogen is a bound state of a proton to a muon (instead of a proton and an electron as in the usual hydrgen).

i) Use Bohr's theory to calculate the energy levels of the muonic hydrogen. What is the energy of the groundstate (in eV)? What is a difference as compared to the usual hydrogen?

ii) What is the wavelength of a photon emitted in a transition between the first excited state and the ground state? Which kind of photon is it (radio, microwave, visible, ultraviolet, X-ray, γ-ray, ...)?

iii) Besides the mass, muons differ from electron by the fact that they "decay" into an electron and two massless, chargeless particles called "neutrinos". The lifetime for the decay is about 2×10^{-6}s. Does the muon have the time to orbit the proton several times before decaying when it is in the ground state of muonic hydrogen?