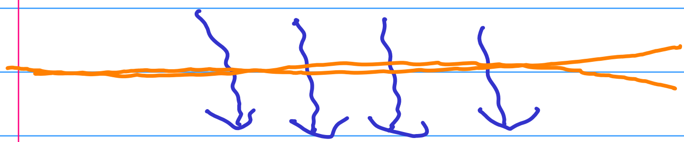
Exercise 2: Addition of angular momentum 04/11/2020



A. Two particles with spin $\frac{1}{2}$

$\hbar = 1 \qquad S_1, \; S_2$

$$S_1^2 |s_1, m_1\rangle = s_1(s_1+1)|s_1, m_1\rangle$$

$$S_{1z} |s_1, m_1\rangle = m_1 |s_1, m_1\rangle$$

$1 \longleftrightarrow 2$

$|s_1 = \frac{1}{2}, m_1 = \frac{1}{2}\rangle \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|s_1 = \frac{1}{2}, m_1 = -\frac{1}{2}\rangle \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\uparrow\rangle \; , \; |\downarrow\rangle$

$$|s_1, m_1\rangle \otimes |s_2, m_2\rangle = |s_1 m_1, s_2 m_2\rangle$$

$|\uparrow\uparrow\rangle = |\frac{1}{2} \; \frac{1}{2}\rangle \otimes |\frac{1}{2} \; \frac{1}{2}\rangle$

$|\uparrow\downarrow\rangle = |\frac{1}{2} \; \frac{1}{2}\rangle \otimes |\frac{1}{2} \; -\frac{1}{2}\rangle$

$|\downarrow\uparrow\rangle = |\frac{1}{2} \; -\frac{1}{2}\rangle \otimes |\frac{1}{2} \; \frac{1}{2}\rangle$

$|\downarrow\downarrow\rangle = |\frac{1}{2} \; -\frac{1}{2}\rangle \otimes |\frac{1}{2} \; -\frac{1}{2}\rangle$

"product basis"

$S = S_1 + S_2$

$[S_x, S_y] = i S_z$

$$S^2 |s, m\rangle = s(s+1)|s, m\rangle$$

$$S_z |s, m\rangle = m |s, m\rangle$$

Eigenproblem for total spin $S = S_1 + S_2$

$[S_1^2, S_z] = 0 = [S_1^2, S_{z_1}]$

$S_z = S_{z1} + S_{z2}$

$S_z|\uparrow\uparrow\rangle = (S_{z1} + S_{z2})|\uparrow\uparrow\rangle = (m_1 + m_2)|\uparrow\uparrow\rangle$
$= m\,|\uparrow\uparrow\rangle = 1\,|\uparrow\uparrow\rangle$

$S_z|\uparrow\downarrow\rangle = 0$
$S_z|\downarrow\uparrow\rangle = 0$
$S_z|\downarrow\downarrow\rangle = -m|\downarrow\downarrow\rangle = -1\,|\downarrow\downarrow\rangle$

$S_{z,ij} = \langle i|S_z|j\rangle$

$|\psi\rangle = \sum_j c_j |j\rangle$

$$S_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\lambda_1 = 1$ $\quad \lambda_2 = 0 \quad \lambda_3 = 0 \quad \lambda_4 = -1$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad v = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$|\alpha|^2 + |\beta|^2 = 1$

Eigenstates of S_z $\qquad |\psi\rangle = \sum_j v_j |j\rangle$

$|\psi_1\rangle = |\uparrow\uparrow\rangle$

$$S^2 = (S_1 + S_2)(S_1 + S_2) = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$$

matrix representation in product basis

$$S^2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$\lambda_1 = 2$ $\lambda_2 = 2$ $\lambda_3 = 2$ $\lambda_4 = 0$

9:42 $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $v_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$|\psi_1\rangle = |\uparrow\uparrow\rangle$ $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ $|\psi_4\rangle =$

$= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$|\psi_3\rangle = |\downarrow\downarrow\rangle$

$S^2 |\psi_{1,2,3}\rangle = 2 |\psi_{1,2,3}\rangle$

$s(s+1)$ $s = 1$

$S_z |\uparrow\uparrow\rangle = 1 \cdot |\uparrow\uparrow\rangle$ $m = 1$

$S_z |\psi_2\rangle = 0$ $m = 0$

$S_z |\psi_3\rangle = -|\downarrow\downarrow\rangle$ $m = -1$

$S^2 |\psi_4\rangle = 0 \Rightarrow$ $s = 0$

$S_z |\psi_4\rangle = 0$ $m = 0$

$$\left.\begin{array}{l} |\psi_1\rangle = |s=1, m=1\rangle \\ |\psi_2\rangle = |s=1, m=0\rangle \\ |\psi_3\rangle = |s=1, m=-1\rangle \end{array}\right\} \text{triplet state}$$

$$|\psi_4\rangle = |s=0, m=0\rangle \} \text{ singlet state}$$

in product basis: $S_1^2, S_2^2, S_{z1}, S_{z1}$

in total spin basis: S^2, S_z, S_1^2, S_2^2

$H \sim \gamma_1 S_1 + \gamma_2 S_2$

$H \sim S_1 \cdot S_2$

B) general case J_1, J_2

$J = J_1 + J_2 \qquad J_z = J_{z1} + J_{z2}$

$J_1^2 |j_1, m_1\rangle \qquad J_{z1} |j_1, m_1\rangle$

$1 \longleftrightarrow 2$

"product basis"

$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1 m_1, j_2 m_2\rangle$

$J_z |j_1, m_1\rangle \otimes |j_2, m_2\rangle = \underbrace{(m_1 + m_2)}_{m} |j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$m = j_1 + j_2 - 2 \qquad m_1 = j_1 \qquad m_2 = j_2 - 2$

$m_1 = j_1 - 1 \qquad m_2 = j_2 - 1$

$m_1 = j_1 - 2 \qquad m_2 = j_2$

$J^2 |j, m\rangle = j(j+1) |j, m\rangle$

$J_z |j, m\rangle = m |j, m\rangle$

$|j_1 - j_2| \leq j \leq j_1 + j_2 \qquad -j < m < +j$

$m \downarrow \qquad \longrightarrow j$

$|j_1+j_2, j_1+j_2\rangle$

$|j_1+j_2, j_1+j_2-1\rangle \qquad |j_1+j_2-1, j_1+j_2-1\rangle \quad \cdots$

$|j_1+j_2, j_1+j_2-2\rangle \qquad |j_1+j_2-1, j_1+j_2-2\rangle \quad \cdots$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$$|j, m\rangle = |j_1+j_2, j_1+j_2\rangle = |j_1\, m_1 = j_1\rangle \otimes |j_2\, m_2 = j_2\rangle$$

$$J_-|j, m\rangle = \sqrt{2j}\,|j, m-1\rangle$$

$$J_-|j_1+j_2, j_1+j_2\rangle = (J_{-1}+J_{-2})|j_1\, m_1=j_1\rangle \otimes |j_2\, m_2=j_2\rangle =$$

$$= \sqrt{2j_1}\,|j_1\, m_1 = j_1-1\rangle \otimes |j_2, m_2 = j_2\rangle$$

$$+ \sqrt{2j_2}\,|j_1\, m_1 = j_1\rangle \otimes |j_2\, m_2 = j_2-1\rangle$$

$$|jm\rangle = \sum_{m_1 m_2}{}' |j_1 m_1, j_2 m_2\rangle \underbrace{\langle j_1 m_1, j_2 m_2 | jm\rangle}_{\text{Clebsch-Gordon}}$$

$$|jm\rangle = \sum_{\substack{m_1 m_2 \\ m_1+m_2=m}} C^{jm}_{j_1 m_1, j_2 m_2}\, |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$j_1, j_2; \quad m_1, m_2$

C: $J = L + S$

$j = \ell \pm \frac{1}{2}$

j	$m_s = \frac{1}{2}$	$m = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\left(\frac{\ell + m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\frac{\ell - m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$
$\ell - \frac{1}{2}$	$-\left(\frac{\ell - m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$	$\left(\frac{\ell + m + \frac{1}{2}}{2\ell+1}\right)^{1/2}$

$\ell = 2 \quad , \quad s = \frac{1}{2}$

$j = 2 + \frac{1}{2} = \frac{5}{2} \qquad -j < m < j$

$j = 2 - \frac{1}{2} = \frac{3}{2}$

$|j, m\rangle$

$$\left|\frac{5}{2}, \frac{3}{2}\right\rangle = \sum_{m_\ell, m_s} C^{j=\frac{5}{2}\, m=\frac{3}{2}}_{\ell m_\ell, s m_s} |\ell\, m_\ell\rangle \otimes |s, m_s\rangle$$

$$m_\ell + m_s = m$$

$$\left|\frac{5}{2}, \frac{3}{2}\right\rangle = \frac{2}{\sqrt{5}} |2, 1\rangle \otimes \left|\frac{1}{2} \frac{1}{2}\right\rangle + \frac{1}{\sqrt{5}} |2\, 2\rangle \left|\frac{1}{2} -\frac{1}{2}\right\rangle$$

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Book to read e.g.:

Ramamurti Shankar, "Quantum mechanics"